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CR-SUBMANIFOLDS OF A COMPLEX SPACE FORM

AUREL BEJANCU, MASAHIRO KON & KENTARO YANO

Dedicated to Professor Buchin Su on his 80th birthday

0. Introduction

The CR-submanifolds of a Kaehlerian manifold have been defined by one of the present authors and studied by him [2], [3] and by B. Y. Chen [4].

The purpose of the present paper is to continue the study of CR-submanifolds, and in particular of those of a complex space form.

In §1 we first recall some fundamental formulas for submanifolds of a Kaehlerian manifold, and in particular for those of a complex space form, and then give the definitions of CR-submanifolds and generic submanifolds in our context. We also include Theorem 1 which seems to be fundamental in the study of CR-submanifolds.

In §2 we study the f-structures which a CR-submanifold and its normal bundle admit. We then prove Theorem 2 which characterizes generic submanifolds with parallel f-structure of a complex space form.

In §3 we derive an integral formula of Simons' type and applying it to prove Theorems 3, 4 and 5.

1. Preliminaries

Let $\bar{M}$ be a complex m-dimensional (real $2m$-dimensional) Kaehlerian manifold with almost complex structure J, and M a real n-dimensional Riemannian manifold isometrically immersed in $\bar{M}$. We denote by $\langle\ ,\ \rangle$ the metric tensor field of $\bar{M}$ as well as that induced on M. Let $\bar{\nabla}$ (resp. ∇) be the operator of covariant differentiation with respect to the Levi-Civita connection in $\bar{M}$ (resp. M). Then the Gauss and Weingarten formulas for M are respectively written as

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X N = -A_N X + D_X N$$

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for any vector fields X, Y tangent to M and any vector field N normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. Both A and B are called the second fundamental forms of M and are related by $\langle A_N X, Y\rangle = \langle B(X, Y), N\rangle$.

For any vector field X tangent to M we put

$$JX = PX + FX, \tag{1.1}$$

where PX is the tangential part of JX, and FX the normal part of JX. Then P is an endomorphism of the tangent bundle $T(M)$ of M, and F is a normal bundle valued 1-form on $T(M)$.

For any vector field N normal to M we put

$$JN = tN + fN, \tag{1.2}$$

where tN is the tangential part of JN, and fN the normal part of JN.

If the ambient manifold $\overline{M}$ is of constant holomorphic sectional curvature c, then $\overline{M}$ is called a complex space form, and will be denoted by $\overline{M}^m(c)$. Thus the Riemannian curvature tensor $\overline{R}$ of $\overline{M}^m(c)$ is given by

$$\overline{R}(X, Y)Z = \tfrac{1}{4}c[\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ]$$

for any vector fields X, Y and Z of $\overline{M}^m(c)$. We denote by R the Riemannian curvature tensor of M. Then we have

$$R(X, Y)Z = \tfrac{1}{4}c[\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle PY, Z\rangle PX - \langle PX, Z\rangle PY + 2\langle X, PY\rangle PZ] + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \tag{1.3}$$

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tfrac{1}{4}c[\langle PY, Z\rangle FX - \langle PX, Z\rangle FY + 2\langle X, PY\rangle FZ] \tag{1.4}$$

for any vector fields X, Y and Z tangent to M.

If the second fundamental form B of M satisfies the classical Codazzi equation $(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)$, then (1.4) implies (cf., [1, p. 434])

Lemma 1. *Let M be an n-dimensional submanifold of a complex space form $\overline{M}^m(c)$, $c \neq 0$. If the second fundamental form of M satisfies the classical Codazzi equation, then M is holomorphic or anti-invariant.*

Definition 1. A submanifold M of a Kaehlerian manifold $\overline{M}$ is called a CR-submanifold of $\overline{M}$ if there exists a differentiable distribution $\mathfrak{D}: x \to \mathfrak{D}_x \subset T_x(M)$ on M satisfying the following conditions:

(i) $\mathfrak{D}$ is holomorphic, i.e., $J\mathfrak{D}_x = \mathfrak{D}_x$ for each $x \in M$, and

(ii) the complementary orthogonal distribution $\mathfrak{D}^{\perp}: x \to \mathfrak{D}_x^{\perp} \subset T_x(M)$ is anti-invariant, i.e., $J\mathfrak{D}_x^{\perp} \subset T_x(M)^{\perp}$ for each $x \in M$.

If dim $\mathfrak{D}_x^\perp = 0$ (resp. dim $\mathfrak{D}_x = 0$) for any $x \in M$, then the *CR*-submanifold is a holomorphic submanifold (resp. anti-invariant submanifold) of $\bar{M}$. If dim $\mathfrak{D}_x^\perp =$ dim $T_x(M)^\perp$ for any $x \in M$, then the *CR*-submanifold is a generic submanifold of M (see [9]). It is clear that every real hypersurface of a Kaehlerian manifold is automatically generic submanifold. A *CR*-submanifold is called a proper *CR*-submanifold if it is neither a holomorphic submanifold nor an anti-invariant submanifold. From Lemma 1 we have

Proposition 1. *Let M be a proper CR-submanifold of a complex space form $\bar{M}^m(c)$. If the second fundamental form of M satisfies the classical Codazzi equation, then $c = 0$.*

A submanifold M is said to be minimal if trace $B = 0$. If $B = 0$ identically, M is called a totally geodesic submanifold.

Definition 2. A *CR*-submanifold M of a Kaehlerian manifold $\bar{M}$ is said to be mixed totally geodesic if $B(X, Y) = 0$ for each $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$.

Lemma 2. *Let M be a CR-submanifold of a Kaehlerian manifold $\bar{M}$. Then M is mixed totally geodesic if and only if one of the following conditions is fulfilled:*

(i) $A_N X \in \mathfrak{D}$ *for any* $X \in \mathfrak{D}$ *and* $N \in T(M)^\perp$,

(ii) $A_N Y \in \mathfrak{D}^\perp$ *for any* $Y \in \mathfrak{D}^\perp$ *and* $N \in T(M)^\perp$.

The integrability of distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$ on a *CR*-submanifold M is characterized by

Theorem 1. *Let M be a CR-submanifold of a Kaehlerian manifold $\bar{M}$. Then we have*

(i) $\mathfrak{D}^\perp$ *is always involutive*, [4],

(ii) $\mathfrak{D}$ *is involutive if and only if the second fundamental form B satisfies* $B(PX, Y) = B(X, PY)$ *for all* $X, Y \in \mathfrak{D}$, [2].

Definition 3. A *CR*-submanifold M is said to be mixed foliate if it is mixed totally geodesic and $B(PX, Y) = B(X, PY)$ for all $X, Y \in \mathfrak{D}$.

Now, let $M^\perp$ be a leaf of anti-invariant distribution $\mathfrak{D}^\perp$ on M. then we have

Proposition 2. *A necessary and sufficient condition for the submanifold $M^\perp$ to be totally geodesic in M is that*

$$B(X, Y) \in fT(M)^\perp \text{ for all } X \in \mathfrak{D}^\perp \text{ and } Y \in \mathfrak{D}.$$

Proof. For any vector fields X and Y tangent to M, (1.1) and Gauss and Weingarten formulas imply

$$tB(X, Y) = (\nabla_X P) Y - A_{FY} X, \tag{1.5}$$

where we have put $(\nabla_X P) Y = \nabla_X PY - P \nabla_X Y$.

Let $X, Z \in \mathfrak{D}^\perp$ and $Y \in \mathfrak{D}$. Then (1.5) implies that

$$\langle P\nabla_X Z, Y\rangle = -\langle A_{FZ}X, Y\rangle = -\langle B(X, Y), FZ\rangle,$$

which proves our assertion.

Corollary 1. *Let M be a mixed totally geodesic CR-submanifold of a Kaehlerian manifold $\bar{M}$. Then each leaf of anti-invariant distribution $\mathfrak{D}^\perp$ is totally geodesic in M.*

Corollary 2. *A generic submanifold M of a Kaehlerian manifold $\bar{M}$ is mixed totally geodesic if and only if each leaf of anti-invariant distribution is totally geodesic in M.*

Lemma 3. *Let M be a mixed foliate CR-submanifold of a Kaehlerian manifold $\bar{M}$. Then we have*

$$A_N P + PA_N = 0$$

for any vector field N normal to M.

Proof. From the assumption we have $B(X, PY) = B(PX, Y)$ for all $X, Y \in \mathfrak{D}$. On the other hand, we obtain $B(X, Y) = 0$ for $X \in \mathfrak{D}$ and $Y \in \mathfrak{D}^\perp$. Moreover, we see that $PX \in \mathfrak{D}$ for any vector field X tangent to M. Consequently we can see that $B(X, PY) = B(PX, Y)$ for any vector fields X, Y tangent to M, from which it follows that $A_N P + PA_N = 0$.

Proposition 3. *If M is a mixed foliate proper CR-submanifold of a complex space form $\bar{M}^m(c)$, then we have $c \leqslant 0$.*

Proof. Let $X, Y \in \mathfrak{D}$ and $Z \in \mathfrak{D}^\perp$. Then we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, \nabla_Y Z) - B(Y, \nabla_X Z).$$

If we take a vector field U normal to M such that $Z = JU = tU$, we obtain that $\nabla_Y Z = -PA_U Y + tD_Y U$. Thus Lemma 3 implies that

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(PY, A_U X) + B(X, A_U PY).$$

Putting $X = PY$ and using (1.4) we see that $2B(PY, A_U PY) = -\frac{1}{2}c\langle PY, PY\rangle U$. Therefore we have

$$0 \leqslant 2\langle A_U PY, A_U PY\rangle = -\tfrac{1}{2}c\langle PY, PY\rangle\langle U, U\rangle, \tag{1.6}$$

which proves our assertion.

Corollary 3. *Let M be a mixed foliate CR-submanifold of a complex space form $\bar{M}^m(c)$. If $c > 0$, then M is a holomorphic submanifold or an anti-invariant submanifold of $\bar{M}^m(c)$.*

2. f-structure

Let M be an n-dimensional CR-submanifold of a complex m-dimensional Kaehlerian manifold $\bar{M}$. Applying J to both sides of (1.1) we have

$$-X = P^2X + tFX,$$

from which it follows that $P^3X + PX = 0$ for any vector field X tangent to M. Thus

$$P^3 + P = 0.$$

On the other hand, the rank of P is equal to dim $\mathfrak{D}_x$ everywhere on M. Consequently, P defines an f-structure on M (see [7]).

Applying J to both sides of (1.2) we obtain that

$$-N = FtN + f^2N,$$

so that $f^3N + fN = 0$ for any vector field N normal to M, and the rank of f is equal to dim $T_x(M)$ − dim $\mathfrak{D}_x$ everywhere on M. Thus f defines an f-structure on the normal bundle of M.

Definition 4. If $\nabla_X P = 0$ for any vector field X tangent to M, then the f-structure P is said to be parallel.

Proposition 4. *Let M be an n-dimensional generic submanifold of a complex m-dimensional Kaehlerian manifold $\bar{M}$. If the f-structure P on M is parallel, then M is locally a Riemannian direct product $M^T \times M^\perp$, where M^T is a totally geodesic complex submanifold of $\bar{M}$ of complex dimension $n - m$, and $M^\perp$ is an anti-invariant submanifold of $\bar{M}$ of real dimension $2m - n$.*

Proof. From the assumption and (1.5) we have $JB(X, Y) = tB(X, Y) = -A_{FY}X$. Thus $JB(X, PY) = 0$ and hence $B(X, PY) = 0$. On the other hand, we see that

$$fB(X, Y) = B(X, PY) + (\nabla_X F)Y. \tag{2.1}$$

Since $f = 0$, we have $(\nabla_X F)Y = -B(X, PY) = 0$.

Let $Y \in \mathfrak{D}^\perp$. Then we have that $P\nabla_X Y = \nabla_X PY - (\nabla_X P)Y = 0$ for any vector field X tangent to M, so that the distribution $\mathfrak{D}^\perp$ is parallel. Similarly, the distribution $\mathfrak{D}$ is also parallel. Consequently, M is locally a Riemannian direct product $M^T \times M^\perp$, where M^T and $M^\perp$ are leaves of $\mathfrak{D}$ and $D^\perp$ respectively. From the constructions, M^T is a complex submanifold of $\bar{M}$, and $M^\perp$ is an anti-invariant submanifold of $\bar{M}$. On the other hand, since $B(X, PY) = 0$ for any vector fields X and Y tangent to M, M^T is totally geodesic in $\bar{M}$. Thus we have our assertion.

Theorem 2. *Let M be an n-dimensional complete generic submanifold of a complex m-dimensional, simply connected complete complex space form $\bar{M}^m(c)$. If the f-structure P on M is parallel, then M is an m-dimensional anti-invariant submanifold of $\bar{M}^m(c)$, or $c = 0$ and M is $C^{n-m} \times M^{2m-n}$ of C^m, where M^{2m-n} is an anti-invariant submanifold of C^m.*

Proof. First of all, we have

$$(\nabla_X B)(Y, PZ) = D_X(B(Y, PZ)) - B(\nabla_X Y, PZ) - B(Y, P\nabla_X Z) = 0,$$

which together with (1.4) implies

$$\tfrac{1}{4}c[\langle PY, PY\rangle FX - \langle PX, PY\rangle FY] = 0.$$

Thus we have $c = 0$ or $P = 0$. If $P = 0$, then M is a real m-dimensional anti-invariant submanifold of $\bar{M}^m(c)$. If $c = 0$, then the ambient manifold $\bar{M}^m(c)$ is a complex number space C^m, and our assertion follows from Proposition 3.

Proposition 5. *Let M be an n-dimensional complex mixed foliate proper generic submanifold of a simply connected complete complex space form $\bar{M}^m(c)$. If $c \geqslant 0$, then $c = 0$ and M is $C^{n-m} \times M^{2m-n}$ of C^m, where M^{2m-n} is an anti-invariant submanifold of C^m.*

Proof. From Proposition 3 we see that $c = 0$ and hence $\bar{M}^m(c) = C^m$. Then (1.6) implies that $A_U X = 0$ for any $X \in \mathfrak{D}$. From this and (1.5) we see that P is parallel. Thus theorem 2 proves our assertion.

3. An integral formula

First of all, we recall the formula of Simons' type for the second fundamental form [6].

Let M be an n-dimensional minimal submanifold of an m-dimensional Riemannian manifold $\bar{M}$. Then the formula of Simons' type for the second fundamental form A of M is written as

$$\nabla^2 A = -A \circ \tilde{A} - \underset{\sim}{A} \circ A + \bar{R}(A) + \bar{R}', \tag{3.1}$$

where we have put $\tilde{A} = {}^tA \circ A$ and $\underset{\sim}{A} = \sum_{a=1}^{m-n} adA_a adA_a$ for a normal frame $\{V_a\}$, $a = 1, \cdots, m - n$, and $A_a = A_{V_a}$. For a frame $\{E_i\}$, $i = 1, \cdots, n$ of M, we put

$$\langle \bar{R}'^N(X), Y\rangle = \sum_{i=1}^{n} \left(\langle(\bar{\nabla}_X \bar{R})(E_i, Y)E_i, N\rangle + \langle(\bar{\nabla}_{E_i} \bar{R})(E_i, X)Y, N\rangle\right) \tag{3.2}$$

for any vector fields X, Y tangent to M and any vector field N normal to M, $\bar{R}$ being the Riemannian curvature tensor of $\bar{M}$. Moreover, we put

$$\begin{aligned}\langle \bar{R}(A)^N(X), Y\rangle &= \sum_{i=1}^{n} \left[2\langle \bar{R}(E_i, Y)B(X, E_i), N\rangle + 2\langle \bar{R}(E_i, X)B(Y, E_i), N\rangle\right.\\ &\quad -\langle A_N X, \bar{R}(E_i, Y)E_i\rangle - \langle A_N Y, \bar{R}(E_i, X)E_i\rangle\\ &\quad \left.+\langle \bar{R}(E_i, B(X, Y))E_i, N\rangle - 2\langle A_N E_i, \bar{R}(E_i, X)Y\rangle\right].\end{aligned} \tag{3.3}$$

In the following, we assume that the ambient manifold $\bar{M}$ is a complex space form $\bar{M}^m(c)$. Since $\bar{M}^m(c)$ is locally symmetric, we have $\bar{R}' = 0$. A straightforward computation gives

$$\begin{aligned}\langle \bar{R}(A)^N(X), Y\rangle &= \tfrac{1}{4}cn\langle A_N X, Y\rangle - \tfrac{1}{2}c\langle A_{FY}X, tN\rangle - \tfrac{1}{2}c\langle A_{FX}Y, tN\rangle \\ &\quad + c\langle fB(X, PY), N\rangle + c\langle fB(Y, PX), N\rangle \\ &\quad + \tfrac{3}{4}c\langle PX, PA_N Y\rangle + \tfrac{3}{4}c\langle PY, PA_N X\rangle - \tfrac{3}{2}c\langle A_N PX, PY\rangle \\ &\quad - \tfrac{1}{2}c\sum_{i=1}^{n}\Big[\langle A_{FE_i}E_i, X\rangle\langle FY, N\rangle + \langle A_{FE_i}E_i, Y\rangle\langle FX, N\rangle \\ &\quad + \tfrac{3}{2}\langle A_{FE_i}X, Y\rangle\langle FE_i, N\rangle\Big].\end{aligned} \tag{3.4}$$

We now prepare some lemmas for later use.

Lemma 4, [9]. *Let M be a generic submanifold of a Kaehlerian manifold $\bar{M}$. Then we have*

$$A_{FX}Y = A_{FY}X$$

for any vector fields X, Y.

Lemma 5. *Let M be a minimal CR-submanifold of a Kaehlerian manifold $\bar{M}$ with involutive distribution $\mathfrak{D}$. Then we have*

$$\sum B(E_\alpha, E_\alpha) = 0$$

for a frame $\{E_\alpha\}$ of $\mathfrak{D}^\perp$.

Proof. We take a frame $\{E_t, E_\alpha\}$ of M such that $\{E_t\}$ and $\{E_\alpha\}$ are frames of $\mathfrak{D}$ and $D^\perp$ respectively. Since $\mathfrak{D}$ is involutive, we have that $\Sigma\, B(E_t, E_t) = 0$ so that $\Sigma\, B(E_\alpha, E_\alpha) = 0$.

We now define a vector field H tangent to M in the following way. Let $\{E_\alpha\}$ be a fame of $\mathfrak{D}^\perp$, and put $H = \Sigma_\alpha A_\alpha E_\alpha$. Then $H = \Sigma_i A_{fe_i} tfe_i$ for any frame $\{e_i\}$ of M and H is independent of the choice of a frame of M.

In the following we assume that M is a generic minimal submanifold of $\bar{M}^m(c)$, $c > 0$, with the second fundamental form B satisfying that $B(PX, Y) = B(X, PY)$ for all X, $Y \in \mathfrak{D}$, which implies that $\mathfrak{D}$ is involutive, and $H \in \mathfrak{D}^\perp$.

From (3.4), using Lemmas 4 and 5 we obtain

$$\langle \bar{R}(A), A\rangle \geqslant \tfrac{1}{4}(n+1)c\|A\|^2. \tag{3.5}$$

On the other hand, we have [6]

$$\langle A \circ \tilde{A}, A\rangle + \langle \underset{\sim}{A} \circ A, A\rangle \leqslant \left(2 - \frac{1}{p}\right)\|A\|^4, \tag{3.6}$$

where p denotes the codimension of M, and $\|A\|$ is the length of the second fundamental form A of M. Thus (3.1), (3.5) and (3.6) imply

$$-\langle \nabla^2 A, A\rangle \leqslant \left(2 - \frac{1}{p}\right)\|A\|^4 - \tfrac{1}{4}(n+1)c\|A\|^2. \tag{3.7}$$

If M is compact orientable, then

$$\int_M \langle \nabla^2 A, A\rangle = -\int_M \langle \nabla A, \nabla A\rangle.$$

Therefore (3.7) implies the following.

Theorem 3. *Let M be an n-dimensional compact orientable generic minimal submanifold of a complex space form $\overline{M}^m(c)$, $c > 0$. If $\mathfrak{D}$ is involutive and $H \in \mathfrak{D}^\perp$, then we have*

$$\int_M \langle \nabla A, \nabla A\rangle \leqslant \int_M \left\{\left(2 - \frac{1}{p}\right)\|A\|^4 - \tfrac{1}{4}(n+1)c\|A\|^2\right\}. \tag{3.8}$$

As the ambient manifold $\overline{M}^m(c)$ we take a complex projective space CP^m with constant holomorphic sectional curvature 4. Then we have

Theorem 4. *Let M be an n-dimensional compact orientable generic minimal submanifold of CP^m with involutive distribution $\mathfrak{D}$. If $H \in \mathfrak{D}^\perp$ and $\|A\|^2 < (n+1)/(2 - 1/p)$, then M is real projective space RP^m and $n = m = p$.*

Proof. From (3.8) we see that M is totally geodesic in CP^m. Thus M is a complex or real projective space (see [1, Lemma 4]). Since M is a generic submanifold, M is a real projective space and anti-invariant in CP^m. Thus we have $n = m = p$ and $\dim \mathfrak{D} = 0$.

Theorem 5. *Let M be an n-dimensional compact orientable generic minimal submanifold of CP^m. If $\mathfrak{D}$ is involutive, $H \in \mathfrak{D}^\perp$, and $\|A\|^2 = (n+1)/(2 - 1/p)$, then M is $S^1 \times S^1$ in CP^2, and $n = m = p = 2$.*

Proof. From the assumption we have $\nabla A = 0$, and M is an anti-invariant submanifold of CP^m, and hence $m = n = p$. Thus our assertion follows from [5, Theorem 3].

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IASI UNIVERSITY, RUMANIA
HIROSAKI UNIVERSITY, JAPAN
TOKYO INSTITUTE OF TECHNOLOGY, JAPAN